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Quadrupole correlations governing the pattern of jet noise

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The effects of convection and refraction dominate the heart-shaped pattern of jet noise. These can be corrected out to yield the small 'basic directivity' of the eddy noise generators. The observed quasi-ellipsoidal pattern was predicted by Ribner (1963, 1964) in a variant of the Lighthill theory postulating isotropic turbulence superposed on a mean shear flow. This had the feature of dealing with the joint effects of the quadrupoles without displaying them individually. The present paper reformulates the theory so as to calculate the relative contributions of the different quadrupole self and cross-correlations to the sound emitted in a given direction. Some minor errors are corrected.

Of the thirty-six possible quadrupole correlations only nine yield distinct non-vanishing contributions to the axisymmetric noise pattern of a round jet. The correlations contribute either $\cos^4\theta$, $\cos^2\theta\sin^2\theta$ or $\sin^4\theta$ directional patterns, where θ is the angle with the jet axis. A separation into parts called 'self noise' (from turbulence alone) and 'shear noise' (jointly from turbulence and mean flow) may be made.

The nine self-noise patterns combine as

$$A\cos^4\theta(1)+A\cos^2\sin^2\theta(\tfrac{7}{8}+\tfrac{7}{8}+\tfrac{1}{8}+\tfrac{1}{8})+A\sin^4\theta(\tfrac{12}{32}+\tfrac{12}{32}+\tfrac{7}{32}+\tfrac{1}{32})$$
$$= A(\cos^2\theta+\sin^2\theta)^2 = A;$$

this is uniform in all directions as it must be, arising from isotropic turbulence. The two non-vanishing shear-noise correlation patterns combine as

$$B\cos^4\theta(1)+B\cos^2\theta\sin^2\theta(\tfrac{1}{2}) = B(\cos^2\theta+\cos^4\theta)/2.$$

The overall 'basic' pattern (self noise plus shear noise) thus has the form $A+B(\cos^2\theta+\cos^4\theta)/2$; this is a slight change from the previous result. The dimensional constants A and B are of comparable magnitude; the pattern in any plane through the jet axis thus resembles an ellipse of modest eccentricity.

Frequency spectra are also discussed, following the earlier work. Since the self noise depends quadratically on turbulent velocity components and the shear noise only linearly, there is a relative shift of the self noise to higher frequencies. This in conjunction with refraction figures in the explanation of the deeper pitch of jet noise radiated at small angles to the axis.

Finally, the predictions are shown to be compatible with recent experimental results.

1. Introduction

Experiments on the refraction of sound by gas jets (Atvars *et al.* 1965; Grande 1966) have given strong and unambiguous support for a very simple model of the directional pattern of jet noise. In brief, the directivity is dominated by the competing effects of convection and refraction. Convection wants to beam the sound waves downstream into a broad fan enveloping the jet, whereas refraction wants to bend the waves out of the jet, weakening the core. The result is a heart-shaped pattern.

The effects of refraction and convection can be corrected out of measured jet noise patterns to yield the *small* 'basic directivity' of the eddy noise generators. Some recent results for this (Grande 1966) and for corresponding 'basic spectra' (MacGregor, unpublished) bear a close relation to recent theoretical results.

The theoretical model (Ribner 1963, and in amplified form 1964) is formulated from Lighthill's (1952, 1954) basic equations, but the development and some of the assumptions are different. One of the results is a prediction of a 'basic directivity' pattern that is quasi-ellipsoidal in each frequency band, which is similar to the patterns derived from measurement.

Another result is the decomposition of the 'basic spectrum' into two primary spectra, one for 'self noise' from the turbulence alone, and the other for 'shear noise' arising jointly from the turbulence and the mean shear flow. The proportions of the two spectra vary markedly with direction, and they peak at different frequencies. Taken in combination (with some assistance from the refraction effect), they explain the observed deeper pitch of noise radiated at small angles with the jet as compared with 90°. This overrides the opposite effects of Doppler shift.

The Ribner model, although incorporating Lighthill's quadrupoles $\rho v_i v_j$, does not display them individually. Instead, only their joint effects are dealt with. This is accomplished by use of the Proudman (1952) formalism, wherein ρv_x^2 governs the combined effect for the **x**-direction of emission. The resulting simplicity effects a great reduction in the volume and labour of analytical work. The price, however, is a loss of detail, and perhaps even a loss of credibility. The thirty-six possible self and cross-correlations of the $\rho v_i v_j$ do not appear separately, and their role is bypassed.

The aim of the present paper is to supply this missing detail by a reversion from the Proudman form to the basic Lighthill formalism. The physical model, as in the earlier work, postulates isotropic turbulence superimposed on a specified mean shear flow. The space–time velocity correlation functions are assumed. All the quadrupole correlations that contribute to the axisymmetric jet noise are evaluated. Their respective directional patterns are combined to produce the joint pattern. The results are further broken down into frequency spectra.

The final results for the directional pattern and spectra merely confirm the general results of Ribner (1963, 1964), with minor corrections. What is new is the display of the relative contributions of various self and cross-correlations to the sound emitted in a given direction. Also new is the comparison, in a final section, of the recent measurements with the earlier predictions.

2. Governing equations

Lighthill (1952, 1954) has shown that the sound pressure radiated to a point $\mathbf{x}$ in the far field by a localized unsteady or turbulent flow is given by

$$p(\mathbf{x},t) = (4\pi c_0^2)^{-1}(x_i x_j/x^3)\int_\infty [\partial^2 T_{ij}/\partial t^2]\,d^3\mathbf{y}, \tag{1}$$

where T_{ij} is a quadrupole strength density,

$$T_{ij} = \rho v_i v_j + \tau_{ij} + (\pi - c_0^2\rho)\,\delta_{ij}, \tag{2}$$

that is normally dominated by the unsteady momentum flux $\rho v_i v_j$, e.g. in a turbulent jet at ambient temperature. Here τ_{ij} is the viscous compressive stress tensor, π is the local pressure, ρ the density, c_0 the ambient speed of sound, v_i the velocity, and $\delta_{ij} = 0$ or 1, as $i \neq j$ or $i = j$; the symbol [ ] designates retarded time, $i, j = 1$, 2 or 3, and repeated indices are summed over. The origin of co-ordinates is taken within the flow.

On retaining only $\rho v_i v_j$ in (2) the sound power $x^2\overline{p^2}/\rho_0 c_0$ radiated in direction (θ, ϕ) in polar co-ordinates (per unit solid angle) may be written

$$P(\theta,\phi) = \frac{x_i x_j x_k x_l}{16\pi^2\rho_0 c_0^5 x^4}\int_\infty\int_\infty \overline{\frac{\partial^2(\rho v_i v_j)}{\partial t^2}\frac{\partial^2(\rho' v_k' v_l')}{\partial t^2}}\,d^3\mathbf{y}'\,d^3\mathbf{y}'', \tag{3}$$

where the first term under the overbar is evaluated at $\mathbf{y}', t'$ and the second term at $\mathbf{y}'', t''$. The generation times t', t'' are suitably retarded relative to the reception time t, which is averaged over at fixed $t'-t''$. Alternatively an ensemble average may be used.

The product average or quadrupole correlation shown with an overbar can be expressed (e.g. Ribner 1962) as a function of the midpoint $\mathbf{y}$ and the separation in space and time (figure 1)

$$\mathbf{y} = \tfrac{1}{2}(\mathbf{y}'+\mathbf{y}''); \quad \mathbf{r} = \mathbf{y}'-\mathbf{y}''; \quad \tau = t'-t''. \tag{4}$$

If the observer distance x is large compared with the flow dimensions

$$c_0\tau \simeq \mathbf{r}.\mathbf{x}/x \tag{5}$$

(Meecham & Ford 1958). A convenient transformation of (3) is then

$$P(\theta,\phi) = \int_\infty P(\theta,\phi,\mathbf{y})\,d^3\mathbf{y}, \tag{6}$$

where

$$P(\theta,\phi,\mathbf{y}) = \frac{\rho_0 x_i x_j x_k x_l}{16\pi^2 c_0^5 x^4}\int_\infty \frac{\partial^4}{\partial\tau^4}\overline{v_i v_j v_k' v_l'}\,d^3\mathbf{r}. \tag{7}$$

Here ρ, ρ' have been approximated by the constant ambient value ρ_0 and the $\partial^4/\partial\tau^4$ operation is to be applied before insertion of relation (5) for τ.

The summations $x_i v_i$, $x_j v_j$, etc., implicit in (7), divided by x, are each merely the component of $\mathbf{v}$ in the direction of $\mathbf{x}$. Thus equation (7) may be re-expressed in the very neat form

$$P(\theta,\phi,\mathbf{y}) = \rho_0(16\pi^2 c_0^5)^{-1}\int_\infty \frac{\partial^4}{\partial\tau^4}\overline{v_x^2 v_x'^2}\,d^3\mathbf{r}, \tag{8}$$

due to Proudman (1952).

The quantity $P(\theta, \phi, \mathbf{y})$ is the acoustic power radiated in direction θ, ϕ (per unit solid angle) from a *unit volume element* at $\mathbf{y}$; it takes the form (7) when based on the Lighthill quadrupole formulation, and the form (8) when based on the Proudman formulation. The Proudman form is by far the simpler: the single correlation $\overline{v_x^2 v_x'^2}$ replaces some thirty-six quadrupole correlations $\overline{v_i v_j v_k' v_l'}$. This simplification was exploited by Ribner (1963, 1964): see § 7 and appendix A.

In what follows, the quadrupole formulation is reduced to nine basic terms that are evaluated explicitly. The equivalence with the corrected results of the Proudman formulation (appendix A) is demonstrated.

3. Inferences from axisymmetry

The sound power emission from a round jet, being axisymmetric, possesses no ϕ-dependence. There is accordingly no change in (6) on taking the ϕ-average,

$$\int P(\mathbf{y}, \theta, \phi)\, d^3\mathbf{y} = \int \{P(\mathbf{y}, \theta, \phi)\}_{\phi\,\mathrm{ave}}\, d^3\mathbf{y}. \tag{9}$$

Thus, although the emission $P(\mathbf{y}, \theta, \phi)$ from an individual volume element of the jet is not in general axisymmetric, only the ϕ-average (or axisymmetric part, as it were) contributes to the overall axisymmetric emission. Physically, the various volume elements of the jet will mutually cancel (on a time-average basis) all deviations from their respective ϕ-average sound power emissions.

The required ϕ-average of (7) may be expressed as

$$\left.\begin{aligned} P(\mathbf{y}, \theta) \equiv \{P(\mathbf{y}, \theta, \phi)\}_{\phi\,\mathrm{ave}} &= \rho_0 (16\pi^2 c_0^5)^{-1} I_{ijkl}\,\mathrm{dir}\,(ijkl), \\ \text{where} \qquad I_{ijkl} &\equiv \int \frac{\partial^4}{\partial \tau^4} \overline{v_i v_j v_k' v_l'}\, d^3\mathbf{r}, \\ \mathrm{dir}\,(ijkl) &\equiv (2\pi)^{-1} \int_0^{2\pi} (x_i x_j x_k x_l / x^4)\, d\phi. \end{aligned}\right\} \tag{10}$$

Since $\quad x_1 = x\cos\theta, \quad x_2 = x\sin\theta\cos\phi, \quad x_3 = x\sin\theta\sin\phi, \qquad (11)$

it is found that dir $(ijkl)$ is non-zero only when $ijkl$ are equal in pairs. The non-zero directional factors are

$$\left.\begin{aligned} \mathrm{dir}\,(1111) &= \cos^4\theta, \\ \mathrm{dir}\,(1212) &= (\tfrac{1}{2})\cos^2\theta\sin^2\theta = \mathrm{dir}\,(1313), \\ \mathrm{dir}\,(2222) &= (\tfrac{3}{8})\sin^4\theta = \mathrm{dir}\,(3333), \\ \mathrm{dir}\,(2323) &= (\tfrac{1}{8})\sin^4\theta, \end{aligned}\right\} \tag{12}$$

together with those obtained by permutations of the indices, which do not alter the value.

It is remarkable that (7) can predict a negative power emission in certain directions for a single quadrupole correlation when $ijkl$ are not equal in pairs. This is not a spurious effect. Instead, it points up the fact that the cross-correlations arise from the average of the square of a sum of quadrupole terms: a single cross-correlation by itself is physically inadmissible. Thus, negative contributions from one cross-correlation are compensated by positive contributions from auto-correlations and other cross-correlations.

The expanded form of (10) reads, upon allowance for redundancies arising from permutations of the indices,

$$P(\mathbf{y},\theta) \sim I_{1111}\cos^4\theta+[I_{1212}+I_{1313}+\tfrac{1}{2}I_{1122}+\tfrac{1}{2}I_{1133}]\,2\cos^2\theta\sin^2\theta + [\tfrac{3}{8}I_{2222}+\tfrac{3}{8}I_{3333}+\tfrac{1}{2}I_{2323}+\tfrac{1}{4}I_{2233}]\sin^4\theta, \quad (13)$$

where $\sim$ implies the proportionality factor $\rho_0(16\pi^2c_0^5)^{-1}$ on the right-hand side. Here, for example, I_{1212} replaces the sum of four equivalent correlation integrals with a weight factor of four. The complete array of weight factors is listed with (15), below.

Part of the formalism leading to (13), and (13) itself, are similar to steps in the work of Kotake & Okazaki (1964), which came to the author's attention after developing the present independent approach. However, the differences (particularly in the later steps) exceed the similarities, and the final results are very divergent.

4. Basic quadrupole correlations

Equation (13) completes the reduction of the directional acoustic power emission from unit volume to an expression involving nine basic correlation integrals. The next step is to set forth expressions for the correlation functions involved. The constituent velocities may be written, e.g.

$$v_i = U\delta_i + u_i, \quad (14)$$

where U is the local mean velocity which is directed along the y_i-axis, and u_i is the contribution of the turbulence, assumed locally homogeneous and isotropic in our model; the special symbol $\delta_i = 1$ if $i = 1$ and is otherwise zero. Thus, the cases where i, j, k or $l = 1$ lead to a multiplicity of terms which are dealt with in appendix B. A number of these contribute nothing to the noise: either they are constant and differentiate out, or they possess a zero integral over $\mathbf{r}$ because of the postulated isotropy. The surviving terms contributing to the I_{ijk} integrals (10) appearing in (13) are

		Shear noise		Self noise	w.f.†
$\overline{v_1v_1v_1'v_1'}\int$	=	$4UU'\overline{u_1u_1'}$	+	$\overline{u_1^2u_1'^2}$	1,
$\overline{v_1v_2v_1'v_2'}\int$	=	$UU'\overline{u_2u_2'}$	+	$\overline{u_1u_2u_1'u_2'}$	4,
$\overline{v_1v_3v_1'v_3'}\int$	=	$UU'\overline{u_3u_3'}$	+	$\overline{u_1u_3u_1'u_3'}$	4,
$\overline{v_1v_1v_2'v_2'}\int$	=			$\overline{u_1^2u_2'^2}$	2,
$\overline{v_1v_1v_3'v_3'}\int$	=			$\overline{u_1^2u_3'^2}$	2,
$\overline{v_2v_2v_2'v_2'}$	=			$\overline{u_2^2u_2'^2}$	1,
$\overline{v_3v_3v_3'v_3'}$	=			$\overline{u_3^2u_3'^2}$	1,
$\overline{v_2v_3v_2'v_3'}$	=			$\overline{u_2u_3u_2'u_3'}$	4,
$\overline{v_2v_2v_3'v_3'}$	=			$\overline{u_2^2u_3'^2}$	2,

(15)

† Permissible permutations of $ijkl$ = weight factors.

where the notation $\int =$ signifies (following Ffowcs Williams & Maidanik 1965) that the right- and left-hand sides make equal contributions to the noise integral: terms making no contribution have been discarded from the right-hand side. The meaning of the designations *shear noise* and *self noise* will now be discussed.

5. Shear noise

Consider the terms like $UU'\overline{u_1 u_1'}$ in (15). For isotropic turbulence it is known (e.g. Batchelor 1953) that the volume integrals of $\overline{u_1 u_1'}$, etc., over $\mathbf{r}$-space must vanish. If, then, the mean-flow factor UU' were constant with $\mathbf{r}$ the cited terms would contribute exactly nothing to the noise integrals. The contribution will be non-zero only when the mean flow is non-uniform (i.e. possesses shear). The noise associated with such source terms is thus called *shear noise.*

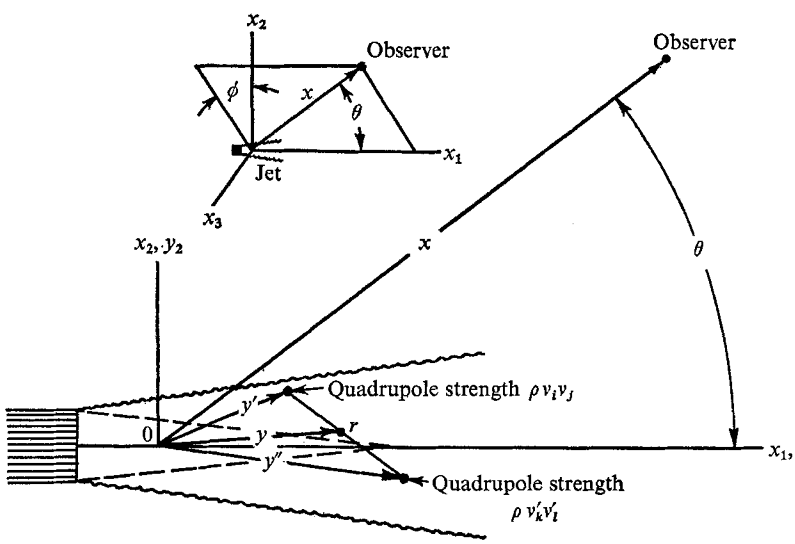

FIGURE 1. Geometry for quadrupole correlations in the jet noise integral.

The terms like $\overline{u_1^2 u_1'^2}$, $\overline{u_1 u_3 u_1' u_3'}$, etc., in (15) contain turbulent velocity components only and are independent of the mean flow. The noise associated with such source terms may be called *self noise.* (Lilley 1958 introduced the phrases shear-amplified noise and self noise; the formalism was quite different so that the correspondence is rather loose.)

We shall evaluate in this section the shear-noise contribution to the directional sound power $P(\mathbf{y}, \theta)$ from a representative unit volume in a jet. For the shear noise the location $\mathbf{y}$ of this volume element is restricted to lie in the annular turbulent mixing region of the jet, at the radius of maximum shear (cf. figure 1). This region has been thought (e.g. Ribner 1958; Dyer 1959) to make the major contribution to the total jet noise. Following Ribner (1963, 1964) the mean flow correlation at $\mathbf{y}$ is taken as

$$U(\mathbf{y}+r_2/2)\,U'(\mathbf{y}-r_2/2) \equiv UU'(\mathbf{r}) \equiv U(\mathbf{y})^2\,e^{-\sigma\pi r_2^2/L^2} \tag{16}$$

in the present model of jet flow.

Equation (10) for $P(\mathbf{y},\theta)$ refers to a stationary reference frame. It will be more convenient, however, to employ a frame moving with the local convection speed $U_c = M_c c_0$, in which the correlations take their simplest form. The Lighthill transformation (1952) allows the $\partial^4/\partial\tau^4$ operation in (10) to be carried out in the moving frame and applies a multiplicative factor $(1-M_c\cos\theta)^{-5}$ (as corrected by Ffowcs Williams 1960) and an exaggerated time delay. We shall reduce this factor to unity by allowing U_c to approach zero so that (10) is formally unaltered, being changed only in interpretation. The effects of convection at finite M_c will be approximated later. With this low-speed stipulation it can be argued that the ratio (eddy size)/(wavelength of sound) is small compared with unity. This implies that the time delay is negligible throughout the volume (approximately the correlation volume) that makes the major contribution to the integral, so we may set $\tau = 0$ therein.

We postulate that the two-point velocity correlations in (15) are factorable into a space factor and a time factor

$$\overline{u_i u_k'} \equiv R_{ik}(\mathbf{r},\tau) = R_{ik}(\mathbf{r})\,g(\tau) \tag{17}$$

in the specified reference frame moving with the local convection speed U_c. The results are somewhat sensitive to the form assumed for the non-dimensional time factor $g(\tau)$, a matter left for discussion in appendix C. For the present $g(\tau)$ will be left unspecified; we shall require, however, the two derivatives

$$\left.\begin{aligned} \omega_f^4(g)^{\mathrm{iv}} &\equiv \left|\frac{\partial^4}{\partial\tau^4}g(\tau)\right|_{\tau=0}, \\ \omega_f^4(g^2)^{\mathrm{iv}} &\equiv \left|\frac{\partial^4}{\partial\tau^4}g^2(\tau)\right|_{\tau=0}, \end{aligned}\right\} \tag{18}$$

together with $g(0) = 1$; ω_f is a characteristic radian frequency $\sim$ (time scale)$^{-1}$, such that $(g)^{\mathrm{iv}}$ and $(g^2)^{\mathrm{iv}}$ are non-dimensional.

The space factor in (17) is taken to be appropriate to homogeneous isotropic turbulence. Thus $R_{ik}(\mathbf{r})$ must have the general form (e.g. Batchelor 1953)

$$R_{ik}(\mathbf{r}) = \overline{u_1^2}[(f+\tfrac{1}{2}rf')\,\delta_{ik} - \tfrac{1}{2}f' r_i r_k/r], \tag{19}$$

where f is some universal function of r. In our model of turbulence this is taken to be

$$f = e^{-\pi^2 r^2/L^2};\quad r^2 = r_1^2+r_2^2+r_3^2, \tag{20}$$

which has been used by Lilley (1958); L is the longitudinal macroscale.

Equations (16) to (20) assemble all the data needed for evaluation of the shear-noise integrals. The required integrals take the form

$$(I_{1111})_{\mathrm{sh}} = 4(g)^{\mathrm{iv}}\,\omega_f^4\int_\infty UU'(\mathbf{r})\,R_{11}(\mathbf{r})\,d^3\mathbf{r}, \tag{21}$$

and two similar integrals $(I_{1212})_{sh}$ and $(I_{1313})_{sh}$ by virtue of (10), (15) and (17) together with the low-speed approximation $\tau \to 0$.

The integration gives

$$(I_{1111})_{sh} = 2\omega_f^4 L^3 U^2 \overline{u_1^2}\,\sigma(1+\sigma)^{-\frac{3}{2}}\,(g)^{\mathrm{iv}} = 4(I_{1313})_{sh}, \tag{22}$$

and $(I_{1212})_{sh}$ vanishes. These results may be inserted into (13), when it is interpreted as

$$P(\mathbf{y},\theta) = \underset{\text{Shear noise}}{P(\mathbf{y},\theta)_{sh}} + \underset{\text{Self noise}}{P(\mathbf{y},\theta)_{se}}. \tag{23}$$

With all the $(I_{ijkl})_{sh}$ equal to zero except $(I_{1111})_{sh}$ and $(I_{1313})_{sh}$ the shear-noise part of (13) takes the form

$$\left.\begin{aligned} P(\mathbf{y},\theta)_{sh} &\sim (I_{1111})_{sh}\cos^4\theta + 2(I_{1313})_{sh}\cos^2\theta\sin^2\theta, \\ P(\mathbf{y},\theta)_{sh} &= B(\cos^4\theta + \tfrac{1}{2}\cos^2\theta\sin^2\theta), \\ &= B\tfrac{1}{2}(\cos^4\theta + \cos^2\theta), \end{aligned}\right\} \tag{24}$$

where the proportionality constant $\rho_0(16\pi^2 c_0^5)^{-1}$ has been absorbed into

$$B \equiv \frac{\rho_0 \omega_f^4 L^3 U^2 \overline{u_1^2}\,\sigma}{8\pi^2 c_0^5 (1+\sigma)^{\frac{3}{2}}}\,(g)^{\mathrm{iv}}, \tag{25}$$

which is constant with respect to θ; its value may vary with source position $\mathbf{y}$ in the jet.

6. Self noise

We proceed to the evaluation of that part of $P(\mathbf{y},\theta)$ arising from turbulence alone (free of cross-coupling with the mean flow) and labelled self noise in (23). The contribution of self noise to I_{ijkl}, by (10) and (15), is

$$(I_{ijkl})_{se} = \int \left[\frac{\partial^4}{\partial\tau^4}\,\overline{(u_i u_j u'_k u'_l)}\right]_{\tau=0} d^3\mathbf{r} \tag{26}$$

under our assumptions. There are nine of these: $ijkl$ = 1111, 2222, 3333; 1122, 2233, 3311; 1212, 1313, 2323. Members in each set of three are equal by isotropy.

We further assume normal joint probability of u_i and u'_k, from which it follows that

$$\begin{aligned} \overline{u_i u_j u'_k u'_l} &= \overline{u_i u_j}\,.\,\overline{u'_k u'_l} + \overline{u_i u'_k}\,.\,\overline{u_j u'_l} + \overline{u_i u'_l}\,.\,\overline{u_j u'_k} \\ &= R_{ij}(0)\,R_{kl}(0) + R_{ik}R_{jl} + R_{il}R_{jk} \end{aligned} \tag{27}$$

(see e.g. Batchelor 1953).

By virtue of the time factor $g(\tau)$ in (17) the integral (26) reduces to

$$(I_{ijkl})_{se} = (g^2)^{\mathrm{iv}}\,\omega_f^4 \int (R_{ik}R_{jl} + R_{il}R_{jk})\,d^3\mathbf{r}, \tag{28}$$

for $\tau = 0$, upon noting that $R_{ij}(0)$, $R_{kl}(0)$ are not τ-dependent.

Upon evaluation of the integrals and insertion into (13), with $P(\mathbf{y},\theta)$ split into shear noise and self noise by (23), there results

$$\left.\begin{aligned} P(\mathbf{y},\theta)_{se} &= A[\cos^4\theta + 2\cos^2\theta\sin^2\theta(\tfrac{7}{16}+\tfrac{7}{16}+\tfrac{1}{16}+\tfrac{1}{16}) \\ &\qquad + \sin^4\theta(\tfrac{12}{32}+\tfrac{12}{32}+\tfrac{7}{32}+\tfrac{1}{32})] \\ &= A(\cos^2\theta + \sin^2\theta)^2 \\ &= A, \end{aligned}\right\} \tag{29}$$

where
$$A = \frac{\sqrt{2}\rho_0 \omega_f^4 (\overline{u_1^2})^2 L^3}{64\pi^2 c_0^5} (g^2)^{\text{iv}} \tag{30}$$

and includes the proportionality constant $\rho_0(16\pi^2 c_0^5)^{-1}$ of (13); like B it is constant with respect to θ, but may vary with source position $\mathbf{y}$ in the jet.

7. Basic broad band noise pattern; associated quadrupole correlations

Upon adding the contributions of self noise (29) and shear noise (24), there results
$$P(\mathbf{y}, \theta) = \underset{\text{Self}}{A} + \underset{\text{Shear}}{B(\cos^4\theta + \cos^2\theta)/2}, \tag{31}$$

for the total noise power from emitted unit volume at $\mathbf{y}$. A and B are of comparable order of magnitude (appendix C).

This is termed a 'basic' pattern, because the normally dominant effects of eddy convection and refraction of the sound by the mean flow are not allowed for; these are dealt with in a later section.

The two parts of the directional pattern (31) are shown in figure 2. One part is a non-directional contribution from the self noise (turbulence alone); the other is a dipole-like contribution from the shear noise (turbulence acting on mean flow). The combined pattern for $A = B$ is a quasi-ellipsoid, with the long axis in the direction of the jet axis.

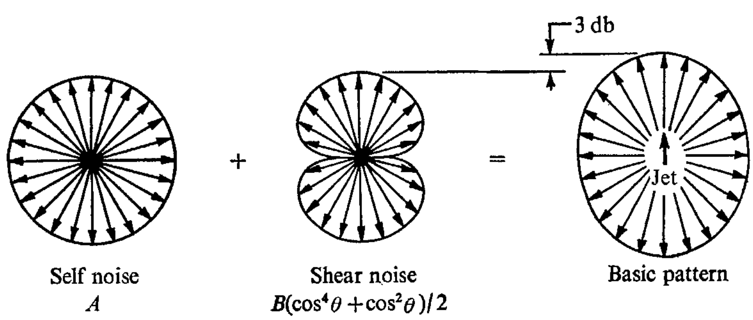

FIGURE 2. Basic pattern of jet noise (before convection and refraction). The self noise is a superposition of nine two-lobe and four-lobe quadrupole patterns; the shear noise is a superposition of two patterns.

Precisely the same pattern (31) is obtained (but far more simply) by use of the Proudman formulation (8) in place of the basic Lighthill formulation (7). The procedure is indicated in Ribner (1963, 1964) as amended by appendix B herein. The earlier analysis led to
$$P(\mathbf{y}, \theta, \phi)_{\phi=0} = A + B\cos^4\theta, \tag{32}$$

which is the basic directional pattern in a certain plane $\phi = 0$; this was mistakenly taken to be the ϕ-average pattern $P(\mathbf{y}, \theta)$. The inappropriate form (32) has figured in some comparisons (e.g. Grande 1966), but fortunately the difference is small.

Let us now consider the basis for the sound pattern (31). It is clear that the isotropic directivity (29),
$$P(\mathbf{y}, \theta)_{\text{se}} = A = \text{constant},$$

of the self noise is a necessary consequence of the isotropy of the turbulence. But it was far from clear *a priori* how the self and cross-correlations of the quadrupole strength densities (the source terms in the noise integrals) would be proportioned to bring this about. Let us examine this point.

The relative contributions of the different quadrupole correlations to the directional factors in the self noise (32) may be discerned with the aid of (13):

$$\left.\begin{aligned} &\cos^4\theta, && \sim 1;\\ &\cos^2\theta\sin^2\theta, && \sim \tfrac{7}{8}+\tfrac{7}{8}+\tfrac{1}{8}+\tfrac{1}{8};\\ &\sin^4\theta, && \sim \tfrac{12}{32}+\tfrac{12}{32}+\tfrac{7}{32}+\tfrac{1}{32}.\end{aligned}\right\} \qquad (33)$$

For the $\cos^4\theta$ term only the self correlation of the self noise part of the longitudinal quadrupole T_{11} contributes. The $\cos^2\theta\sin^2\theta$ term depends on the self correlations of two lateral quadrupoles T_{12} and T_{13} and two cross-correlations $\overline{(T_{11}T'_{22})}_{\text{se}}$, $\overline{(T_{11}T'_{33})}_{\text{se}}$ of longitudinal quadrupoles, in the proportions 7:7:1:1. Finally, the $\sin^4\theta$ term depends on the self correlations of two longitudinal quadrupoles T_{22} and T_{33} and one lateral quadrupole T_{23}, and one cross-correlation $\overline{(T_{22}T'_{33})}_{\text{se}}$ of longitudinal quadrupoles, in the proportions 12:12:7:1. Note that the nine correlations cited here represent through the permutations of the indices a much larger number of correlations, which are not distinct.

The right-hand sides of (33) add up to 1, 2 and 1, respectively, giving the pattern $\cos^4\theta+2\cos^2\theta\sin^2\theta+\sin^4\theta$, which equals unity. Therefore, the inference that the self noise must be independent of θ is confirmed by the detailed examination.

Turning now from the self noise to the shear noise (24), we may make a similar interpretation. The effective quadrupole correlations $\overline{(T_{11}T'_{11})}_{\text{sh}}$, $\overline{(T_{13}T'_{13})}_{\text{sh}}$ contributing to the directional factors $\cos^4\theta$ and $\cos^2\theta\sin^2\theta$ are in the ratio $1:\frac{1}{2}$ respectively. A third quadrupole correlation $\overline{(T_{12}T'_{12})}_{\text{sh}}$ has a zero integral or mean. That is, for the $\cos^4\theta$ term only the self correlation of the longitudinal quadrupole ρ_0Uu_1 contributes. For the $\cos^2\theta\sin^2\theta$ only the self correlation of the lateral quadrupole ρ_0Uu_3 contributes. The form of these shear-noise quadrupoles deserves special note; they involve a cross-coupling between the mean flow U and the turbulence u_i. They correspond physically to a transport of mean flow momentum by the turbulent fluctuations.

The reasons for the basic broadband noise pattern,

$$\text{Intensity} \sim \underset{\text{Self}}{A} + B(\cos^4\theta+\underset{\text{Shear}}{\cos^2\theta})/2,$$

may be summed up. By 'basic' we mean convection and refraction effects are excluded, as mentioned earlier. The non-directional self noise A results from the joint contribution of nine quadrupole correlations having $\cos^4\theta$, $\cos^2\theta\sin^2\theta$ or $\sin^4\theta$ directionality. The proportions are such that they combine to the non-directional pattern. The directional shear noise, $B(\cos^4\theta+\cos^2\theta)/2$, results from two quadrupole correlations having $\cos^4\theta$ and $\cos^2\theta\sin^2\theta$ respective directionalities.

Thus, in the present model (taking due account of the shear in the mean flow) the noise pattern is contributed to by a number of quadrupoles, both longitudinal and lateral. No single quadrupole is dominant. The combination leads to a definite resultant directionality, equation (31), for the noise from unit volume in the mixing region of a jet.

The cross-correlations are seen to be very small: their relative strengths are $\frac{1}{7}$ and $\frac{1}{12}$. Thus, the emissions can be added in the mean square with little error, ignoring all cross-coupling between quadrupoles. The quadrupoles behave very nearly as if they were statistically independent.

8. Effects of convection and refraction

The analysis so far has been postulated on a negligibly small flow Mach number, so that the effects of source convection do not appear. The result obtained for the basic broadband directional pattern radiated from unit volume at **y**,

$$P(\mathbf{y},\theta) = A + B(\cos^4\theta + \cos^2\theta)/2, \tag{31}$$

may be generalized to allow for an eddy convection speed $U_c = M_c c_0$ ($\approx U_j/2$ in the mixing region) in the form

$$P_c(\mathbf{y},\theta) = \underset{\substack{\text{Convection}\\ \text{factor}}}{C^{-5}}[\underset{\text{Basic pattern}}{A + B(\cos^4\theta + \cos^2\theta)/2}], \tag{34}$$

where $$C = [(1-M_c\cos\theta)^2 + \omega_f^2 L^2/\pi c_0^2]^{\frac{1}{2}} = [(1-M_c\cos\theta)^2 + \alpha^2 M_c^2]^{\frac{1}{2}} \tag{35}$$

is a refinement of Lighthill's well-known factor, $1-M_c\cos\theta$ (Ffowcs Williams 1963; Ribner 1962). The basis for this generalization is developed in Ribner (1964) in connexion with the pattern (32). (The later developments in § 9 may also be traced to this same reference.) Here ω_f and L are a characteristic frequency and scale of the turbulence (not necessarily those in the earlier analysis), and the equality of the two forms of (35) defines the non-dimensional parameter α. The empirical value $\alpha = 0{\cdot}55$ gives good agreement with experiment for turbojets (see § 10).

These broad- and narrow-band equations (which follow) omit the powerful effect of refraction of the sound by the jet mean velocity field. This dominates for $|\theta| \leqslant 30°$ for all but the lowest frequencies (cf. figure 3). The refraction effect has been evaluated quantitatively by means of experiments with a pure-tone point source placed in an air jet (Atvars *et al.* 1965; Grande 1966). The result may be expressed as a

$$\text{Refraction factor} = \left(\frac{\text{Intensity at }\theta°}{\text{Intensity at }90°}\right)_f, \tag{36}$$

for frequency f, which multiplies the narrow-band version of (34) (see below). Thus, experimental narrow-band sound measurements should in effect be multiplied by the inverse of (36), to eliminate the refraction effect, before comparison with theory.

The effects of convection and refraction are illustrated in figure 4; this shows the sequence as the basic pattern is modified first by convection and next by refraction to produce the final directional pattern of jet noise. The figure refers to the sound power radiated from unit volume in the mixing region, but the spatial pattern may be taken as typical for the jet as a whole.

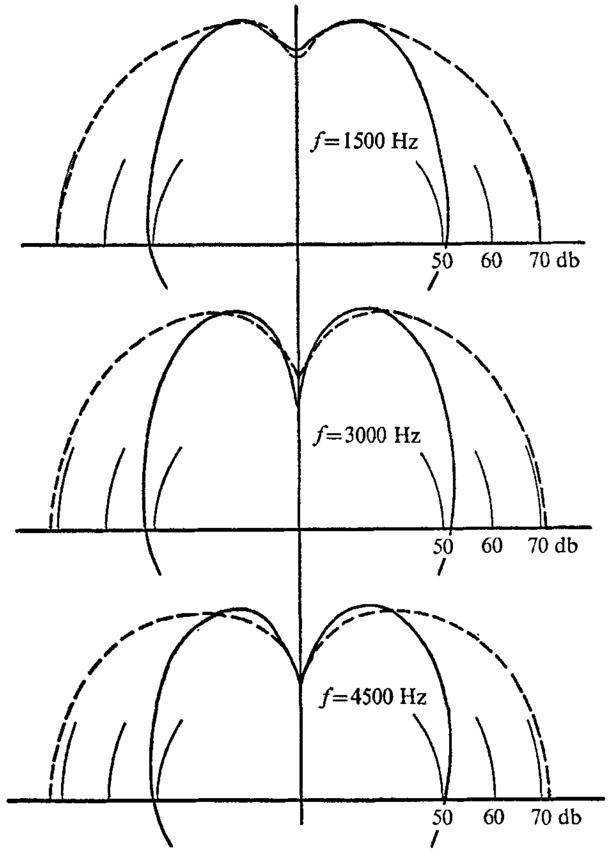

FIGURE 3. Match of noise patterns to indicate refraction-dominated zone of filtered jet noise ($M = 0{\cdot}9$). The pure tone pattern from an oscillator-driven point source placed in the jet defines the refraction effect. — — —, pure tone; ———, filtered jet noise.

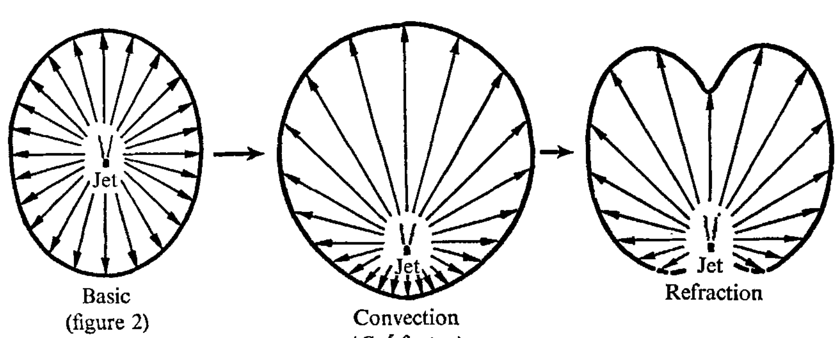

FIGURE 4. Jet noise as a basic pattern (not very directional), which is powerfully modified by convection of the eddy sources and refraction of the sound waves by the mean flow.

9. Spectra and narrow band noise patterns

The constants A and B in the basic broad band spectrum (31) radiated from unit volume may be decomposed into their spectral components (Ribner 1963, 1964)

$$\underset{\text{Self noise}}{A = \int_0^\infty a_1(f)\,df,} \quad \underset{\text{Shear noise}}{B = \int_0^\infty b_1(f)\,df.} \tag{37}$$

The spectrum $a_1(f)$ is predicted on this model to peak substantially above the spectrum $b_1(f)$, because the turbulence velocities appear quadratically in the self-noise terms as against linearly in the shear-noise terms.

Thus, consider a Fourier component $e^{i\omega t}$ in u; this appears as $e^{i2\omega t}$ in u^2; the frequency is doubled. The spectrum as a whole is not, however, shifted to double frequency, because the amplitudes are differentially altered, and sum and difference frequencies are introduced. More precisely, the spectrum is the Fourier cosine transform of the correlation time factor $g(\tau)$ for shear noise and of $g^2(\tau)$ for self noise. For example, the form $g(\tau) = e^{-\omega_f^2\tau^2}$ yields $g^2(\tau) = e^{-2\omega_f^2\tau^2}$, which implies the self-noise spectrum peak is shifted above the shear noise peak by a factor $\sqrt{2}$. Other assumptions (see appendix C) would yield somewhat different shifts.

These spectra are generalized to allow for eddy convection by writing

$$A_c = C^{-5}\int a_1(Cf)\,d(Cf), \quad B_c = C^{-5}\int b_1(Cf)\,d(Cf), \tag{38}$$

where the reception frequency f now incorporates an 'effective' Doppler shift C^{-1} relative to the source frequency Cf. This effective Doppler shift is taken as the shift of the spectrum *peak*, which is less than the true Doppler shift $(1-M_c\cos\theta)^{-1}$ of the constituent lines because of a spectrum distortion. The overall convective amplification C^{-5} contained in A_c and B_c can be seen to consist of an amplification C^{-4} plus a 'Doppler' shift C^{-1} of spectral elements. These points are elaborated in Ribner (1964).

If these narrow sharply peaked spectra are summed for all the radiating volume elements of a jet the overall spectrum is obtained. Symbolically,

$$\left.\begin{aligned} a(Cf) &= \int_{\text{jet}} a_1(Cf)\,d^3\mathbf{y}, \\ b(Cf) &= \int_{\text{jet}} b_1(Cf)\,d^3\mathbf{y}. \end{aligned}\right\} \tag{39}$$

The narrow elementary spectra $a_1(Cf)$, $b_1(Cf)$ summed over in (39) and $a(Cf)$, $b(Cf)$ peak at progressively lower frequencies as the source distance from the nozzle increases. Thus $a(Cf)$ and $b(Cf)$ cover a broad frequency band.

To summarize, the basic broad-band directional pattern from unit volume,

$$P(\mathbf{y},\theta) = \underset{\text{Self}}{A} + \underset{\text{Shear}}{B(\cos^4\theta + \cos^2\theta)/2}, \tag{31}$$

is effectively the integral over frequency of the narrow-band pattern (or spectral density for given θ)

$$P(\mathbf{y},\theta,f) = \underset{\text{Self}}{a_1(f)} + \underset{\text{Shear}}{b_1(f)\ (\cos^4\theta + \cos^2\theta)/2}. \tag{40}$$

Allowance for eddy convection at Mach number M_c generalizes this to

$$P_c(\mathbf{y},\theta,f) = C^{-4}[\underset{\text{Self}}{a_1(Cf)} + \underset{\text{Shear}}{b_1(Cf)\,(\cos^4\theta + \cos^2\theta)/2}]. \tag{41}$$

Integration over source-position $\mathbf{y}$ yields the narrow-band pattern (or spectral density for given θ) emitted by the jet as a whole as

$$P_c(\theta,f) = C^{-4}[\underset{\text{Self}}{a(Cf)} + \underset{\text{Shear}}{b(Cf)\,(\cos^4\theta + \cos^2\theta)/2}]. \tag{42}$$

The general forms of the component spectrum functions, $a(Cf)$ and $b(Cf)$, have been discussed by Ribner (1964), based on the work of Powell (1958) ('f^2 and f^{-2} laws').

A simplifying assumption in the foregoing must be pointed out. In the step from (41) to (42) (the integration over the jet), it is implicit that the self-noise and shear-noise directivities derived for the mixing region apply without change in the developed jet. In addition, the convection factor C is to be taken constant and associated with the constant convection Mach number $M_c \simeq (\frac{1}{2})\,U_j/c$ along the annular central surface of the mixing region; the decay of M_c in the developed jet is neglected. These assumptions are made for expediency and justified in part on the ground that the bulk of the noise originates from the mixing and transition regions. The remainder of the noise (essentially the low frequency part of the spectrum) originates from the developed jet and for this the assumptions are clearly faulty.

We content outselves herein with attributing the directivity in the general form (42) to theory, with the spectral forms of $a(Cf)$ and $b(Cf)$ being only loosely specified. A closer specification cannot be made with confidence in view of the oversimplifications in the theory, e.g. those of the last paragraph together with the assumption of isotropy in the turbulence. The integrals of $a(Cf)$ and $b(Cf)$ over frequency are predicted to be of the same order, so that $A \simeq B$ in (31) (appendix C). Further, the peak of the $a(Cf)$ spectrum (self noise) should lie substantially above the peak of the $b(Cf)$ spectrum (shear noise). There are various uncertainties here, so that the theory must remain only qualitative as to this spectrum shift and the relative magnitudes of A and B.

The present viewpoint is seen to represent a relaxation of the over-restrictive mathematical model given in Ribner (1964), with correction of some errors (appendices A and C). The major features are retained but some flexibility is allowed as to the details.

10. Comparison with experiment

The *broad band* pattern (31), as modified to allow for the source convection, has been given as (34):

$$P_c(\mathbf{y},\theta) = \underset{\substack{\text{Convection}\\\text{factor}}}{C^{-5}}[\underset{\text{Basic pattern}}{A + B(\cos^4\theta + \cos^2\theta)/2}].$$

This refers to the sound power radiated from unit volume in the mixing region, but the spatial pattern may be taken as typical for the jet as a whole. This pattern with $A = B$ and α (in C, (34)) taken as 0·55 for best fit is compared with measurements for several turbojets in figure 5, adapted from Pietrasanta (1956) and Ribner (1963, 1964). The agreement is quite good over the wide range from $\theta = 40°$ to 180°, covering a 200-fold variation (23 db) in intensity. The failure below 40° is due to refraction (figure 3).

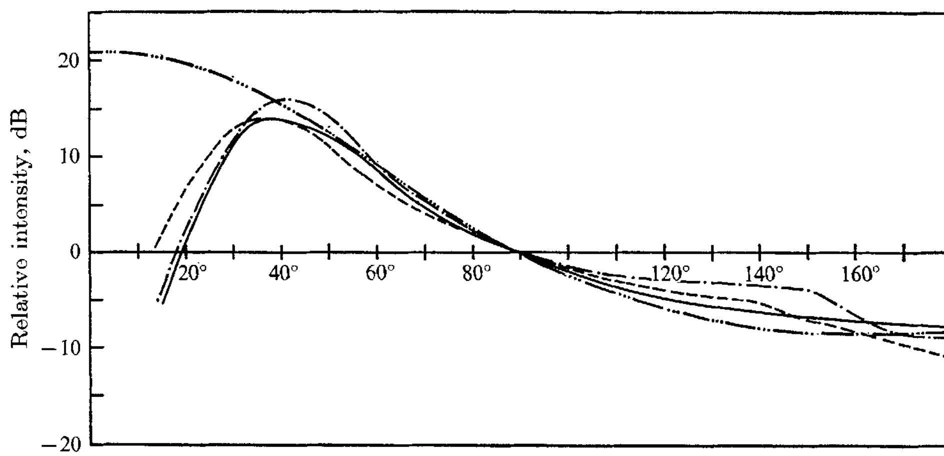

FIGURE 5. Jet noise patterns for turbojets. Comparison of experiment and theory with $A = B$ and α^2 (in C) = 0·3. Pietrasanta: — - —, M_c = 0·76 (J33-A-20 engine); ———, M_c = 0·78 (J 34-WE-34 engine); – – –, M_c = 0·84 (J 48-P-8 engine). —- - - —, M_c = 0·82. Theory: $[(1-M_c\cos\theta)^2+0{\cdot}3M_c^2]^{-\frac{5}{2}}.[1+(\cos^4+\cos^2\theta)/2]$.

The powerful convection factor C^{-5} together with the refraction jointly dominate the pattern at these turbojet flow speeds. The basic directional pattern $1+(\cos^4\theta+\cos^2\theta)/2$ accounts for only 3 db of the 23 db variation. Thus, although the final pattern is markedly directional, the basic pattern (which omits convection and refraction effects) is only weakly directional.

The *narrow band* pattern or directional spectrum (42) reads

$$P_c(\theta,f) = C^{-4}[a(Cf)+b(Cf)(\cos^4\theta+\cos^2\theta)/2].$$

This corresponds to the passage of the broad band radiation in direction θ through a filter of unit band width centred at frequency f. The possibility of testing an equation of this kind (based on (32) in place of (31)) motivated the work of MacGregor (unpublished); he has made comparisons of the theory with spectral measurements on a $\frac{3}{4}$ in. air jet in the Institute for Aerospace Studies anechoic chamber.

I am indebted to MacGregor for figures 6 and 7 herein. Figure 6 is a two-component spectrum obtained by fitting the theoretical model (42) to his experimental data measured at $\theta = 45°$ and 90°. We identify the $a(Cf)$ peak with the self noise and the $b(Cf)$ peak with shear noise. The shift of the self-noise peak well above the shear-noise peak is in general agreement with the theoretical argument given earlier.

Figure 7 shows the narrow band 'basic' directional pattern at $f = 1500$ Hz. The dashed curve is obtained from (42) with the empirical $a(Cf)$ and $b(Cf)$ of figure 6, and with C^{-4} omitted. The solid curve is direct experimental data after correction for refraction (Grande 1966) and for convection by multiplication by C^4.

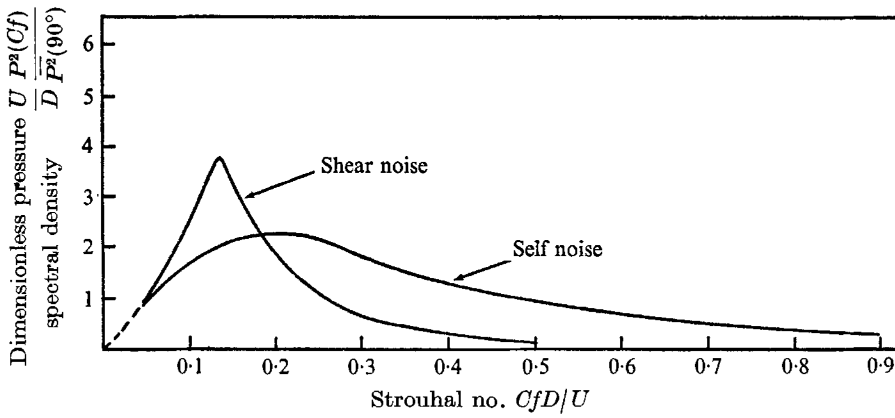

FIGURE 6. Two-componental spectrum obtained by fitting theoretical equation (45) to experiment at $\theta = 45°$ and 90°. Dominance of the shear noise by low frequencies and the self noise by higher frequencies is shown.

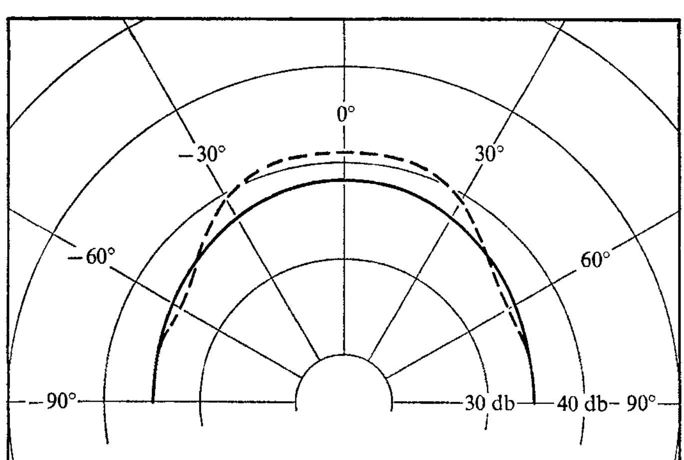

FIGURE 7. 'Basic' directional pattern of jet noise in narrow frequency band at 1500 c/s. Comparison of experiment with theory (curves of figure 6) adjusted for two-point fit. ———, $a(Cf)+b(Cf)(\cos^4\theta+\cos^2\theta)/2$; – – –, experiment. Mach no. = 0·5; $Cf = 1500$ c/s; $(CfD)/U = 0{\cdot}17$.

It is clear that (42) has been used essentially as an interpolation–extrapolation scheme, the two curves having been matched at 45° and 90°. One can conclude, however, that the $a(Cf)+b(Cf)(\cos^4\theta+\cos^2\theta)/2$ form does not deviate markedly from the general shape f the basic experimental pattern. Similar, less distorted patterns were found exp rimentally (with use of reduction methods more loosely defined by theory) by Grande (1966); the deviation from the theoretical quasi-ellipsoidal pattern was noticeably less than in figure 7.

The theoretical predictions for spectra and directivity may also be compared with recent results obtained by Chu (1966) from hot-wire measurements in a jet.

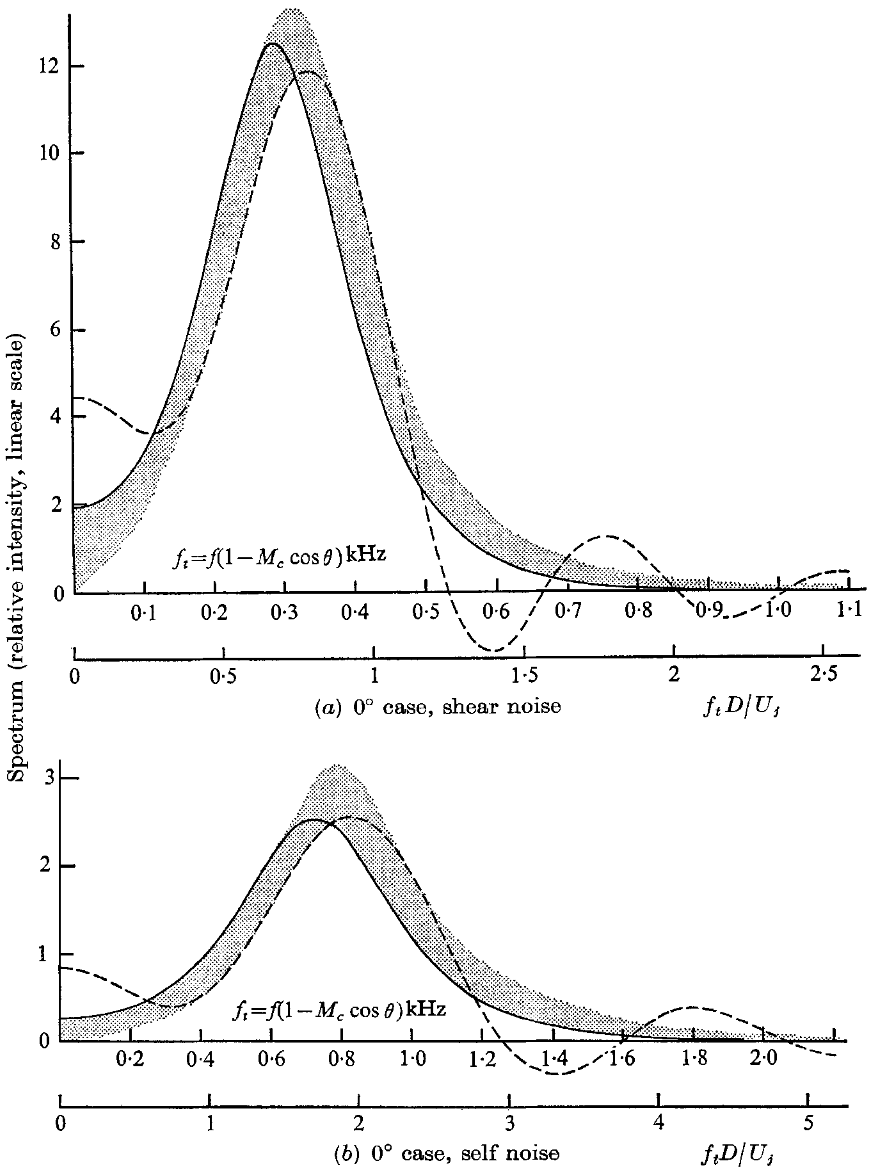

FIGURE 8. Spectra of self noise and shear noise emitted from unit volume of a jet as computed by Chu (1966) from hot-wire measurements of turbulence correlations. ———, Fourier cosine transformation of $A\,\mathrm{sech}\,(b\tau)\cos(c\tau)$; – – –, numerical FCT of fourth derivative curve up to third zero crossing. Shaded area, uncertainty band. 45° and 60° cases will have approximately the same shapes and peaks except for relative intensities. $M_c \simeq 0{\cdot}08$.

Chu measured in great detail the space–time correlations of turbulent velocity figuring in the Lighthill integral in the Proudman form (8). He was able to evaluate numerically a development of this integral and also its Fourier transform to obtain the sound pressure auto-correlation and spectrum. This is a major advance, diminished perhaps by uncertainties in the evaluation of a fourth derivative of an experimental curve.

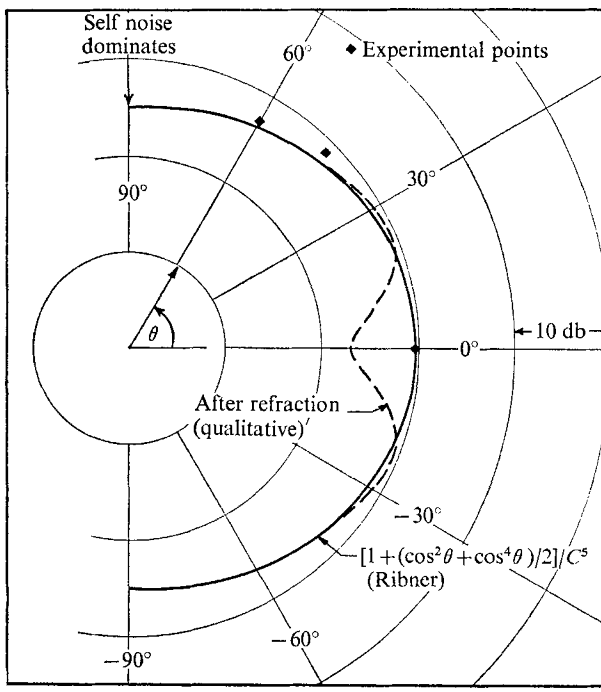

FIGURE 9. Broad band noise emitted from unit volume of a jet in three directions, as computed by Chu (1966) from hot-wire measurements of turbulence correlations.

Chu's results are a prediction of the sound radiated from unit volume (and in three specific directions 0°, 45°, 60°) based on measurements of turbulence within the unit volume. The spectrum he obtained (broken into its two constituents of self noise and shear noise) is shown for the $\theta = 0°$ case in figure 8. The shapes are quite similar to those predicted by the theoretical model (Ribner 1964, figure 16) for $a_1(f)$. In this case the separation of the peaks is even greater than the one octave predicted from the oversimplified argument based on a single Fourier component, $[(e^{i\omega t})^2 = e^{i2\omega t}]$.

The corresponding broad band pattern obtained by Chu is shown in figure 9. This is compared with the pattern (34),

$$P_c(\mathbf{y}, \theta) = C^{-5}[A + B(\cos^4\theta + \cos^2\theta)/2],$$

predicted herein as being emitted from unit volume, with A set equal to B. The agreement of the directionality is quite good.

Since Chu used a very low speed jet (142 ft./sec) the convection factor C^{-5} is near unity. Thus the bracketed 'basic' directional pattern accounts for most (3 db) of the small variation between 0° and 90°. Here again, the quasi-ellipsoidal nature of this basic pattern is evident.

11. Concluding remarks

Since carrying out the present analysis, the author has learned of two other efforts to deal quantitatively with the various quadrupole correlations: Kotake & Okazaki (1964) and Jones (1967).

It is noted that Kotake & Okazaki deal in effect with the self noise only. The shear noise term is dropped on the basis of an order-of-magnitude estimate in the transition from their equation (8) to their equation (9). They seek to explain all of the jet noise properties, including what are now recognized as clearly refractive effects, in terms of the basic directivity of this self noise.

The work of Jones resembles the present study in attempting to tabulate all of the self and cross-correlations contributing to the self noise and the shear noise. The assumptions and the analytical framework are, however, quite different, and the results are not readily compared.

The present formalism may be compared in one respect with the Lighthill (1954) model, which suggests that the single T_{12} lateral quadrupole dominates the shear noise. In his view, '...the most important term in the rate of change of momentum flux $[\partial/\partial t(T_{ij})]$ is the product of the pressure and rate of strain...The higher frequency sound from the heavily sheared mixing region close to the orifice of a jet is found to be of this [single lateral quadrupole] character'. The axisymmetric part of the associated directional pattern has a $\cos^2\theta\sin^2\theta$ directionality. This looks like a four-leaf clover and, even when compounded with self noise, bears little resemblance to the quasi-ellipsoidal pattern deduced herein, either from theory or experiment (figure 7).

However, Csanady (1966) has made a case for supplementing the T_{12} quadrupole with T_{11} and T_{13} quadrupoles in a modified Lighthill model. The details have been made more explicit by Krishnappa (1968). The effect of the T_{11} is to yield, with the added self noise, a fuller pattern more nearly resembling our quasi-ellipsoid.

The Csanady extension, by so modifying the four-leaf-clover pattern, would resolve two divergent views of the dominant directional features of jet noise. On the earlier view, the basic pattern is a four-leaf clover that is distorted by convection to the shape of a butterfly. The partial resemblance at the higher frequencies to the observed heart-shaped pattern of jet noise is stressed. On the present view (Ribner 1963, 1964), convection distorts a basic quasi-ellipsoidal pattern; then refraction (figure 3), omitted in the earlier view, provides the cleft in the final heart-shaped pattern.

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Appendix A. Proudman formulation: correction of Ribner (1963, 1964)

The directional pattern (32) is the radiation pattern from a volume element of a jet evaluated in the plane $\phi = 0$ containing the element and the jet axis. This was obtained in Ribner (1963, 1964) on the basis of the very simple Proudman formulation (8); the result was confused with the ϕ-average. In what follows the procedure is generalized to obtain the radiation pattern in a plane through the jet axis making some arbitrary angle ϕ with the cited plane. Then this pattern is averaged over ϕ to obtain the axisymmetric pattern (31). In addition, the result is generalized to depend on an arbitrary time factor $g(\tau)$ in the velocity correlation in place of the factor $e^{-\omega_f|\tau|}$, which is mathematically unacceptable (appendix C).

Paralleling the procedures herein, but bypassing many of them, Ribner (1964) shows that the integral in (8) is proportional to

$$2(g^2)^{\text{iv}}\,\omega_f^4\int_\infty R_{11}^2(\mathbf{r})\,d^3\mathbf{r}+\tfrac{1}{4}(g)^{\text{iv}}\,\omega_f^4\cos^2\theta\int_\infty UU'(\mathbf{r})\,R_{11}(\mathbf{r})\,d^3\mathbf{r}, \qquad \text{(A 1)}$$

where u_1 and the 1-axis have been chosen parallel to the vector $\mathbf{x}$ drawn from the origin to the observation point. The proportionality factor is $\rho_0(\pi^2c_0^5)^{-1}$, which corrects a misprint.

It will be necessary to re-express (A 1) relative to the present reference frame, in which the 1-axis is parallel to the mean jet flow $\mathbf{U}$. Because of the assumed isotropy of $\mathbf{u}$, R_{11}^2 in the first integral can be retained without change of form, since its volume integral will be unaltered. The first integral is then

$$\text{Self-noise term} = 2^{-4}(g^2)^{\text{iv}}\,\omega_f^4 2^{-\frac{3}{2}}(\overline{u_1^2})^2L^3. \qquad \text{(A 2)}$$

This is a constant, and is unaltered upon averaging over ϕ.

For the second integral in (A 1) the transformation is taken in two steps. First, a rotation is made through an angle θ in the plane containing $\mathbf{x}$ and $\mathbf{U}$ to bring the 1-axis into alignment with $\mathbf{U}$. Call this the r_1', r_2', r_3' frame. Next, a rotation is made about the 1-axis such that the new r_2''-axis makes an angle ϕ with the $\mathbf{x}, \mathbf{U}$ plane. Call this the r_1'', r_2'', r_3'' frame. The connexion between co-ordinates in this and the initial r_1, r_2, r_3 frame is

$$\left.\begin{aligned} r_1 &= r_1''\cos\theta + r_2''\sin\theta\cos\phi - r_3''\sin\theta\sin\phi,\\ r_2 &= -r_1''\sin\theta + r_2''\cos\theta\sin\phi - r_3''\cos\theta\sin\phi,\\ r_3 &= \phantom{-r_1''\sin\theta+{}} r_2''\sin\phi + r_3''\cos\phi. \end{aligned}\right\} \qquad \text{(A 3)}$$

These are to be inserted in R_{11} of (28) and (21)

$$R_{11}(\mathbf{r}) = \overline{u_1^2}[1-(\pi r^2/L^2)+(\pi r_1^2/L^2)]\,e^{-\pi r^2/L^2}. \qquad \text{(A 4)}$$

In doing this we drop the ″, so that the final r_1, r_2, r_3 frame is identified with the r_1, r_2, r_3 frame of the present paper. The result, with the definitions

$$\frac{\pi r_1^2}{L^2} = s_1^2, \quad \frac{\pi r_2^2}{L^2} = s_2^2, \quad \frac{\pi r_3^2}{L^2} = s_3^2, \qquad \text{(A 5)}$$

is $$R_{11}(\mathbf{r}) = \overline{u_1^2}[1-s^2+s_1^2\cos^2\theta+s_2^2\sin^2\theta\cos^2\phi+s_3^2\sin^2\theta\sin^2\phi + \text{cross-product terms}]\exp(-s_1^2-s_2^2-s_3^2). \quad \text{(A 6)}$$

Upon combining (A 6) and (18) the second integral of (A 1) may now be evaluated. The cross-product terms, being odd, will integrate to zero. The integration yields:

$$\text{Shear-noise term} = \frac{\omega_f^4 U^2 \overline{u_1^2}\sigma L^3}{8(1+\sigma)^{\frac{3}{2}}}(\cos^4\theta+\cos^2\theta\sin^2\theta\sin^2\phi)\,(g)^{\text{iv}}. \quad \text{(A 7)}$$

This reduces to equation (9.14) of Ribner (1964) in the plane $\phi = 0$, as it should, when $g(\tau) = e^{-\omega_f\tau}$.

Now, if we average over all azimuth angles ϕ describing the orientation of the volume element around the jet mixing region, $\sin^2\phi$ averages to $\frac{1}{2}$, and the ϕ-average is, after reduction,

$$\text{Shear-noise term} = \frac{\omega_f^4 U^2 \overline{u_1^2}\sigma L^3}{8(1+\sigma)^{\frac{3}{2}}}\left(\frac{\cos^4\theta+\cos^2\theta}{2}\right)(g)^{\text{iv}}. \quad \text{(A 8)}$$

The sum of (A 2) and (A 8) is equivalent to (31).

Appendix B. Nonradiating and radiating quadrupole correlations

The general quadrupole correlation

$$\overline{v_i v_j v_k' v_l'} = \overline{(U\delta_i+u_i)(U\delta_j+u_j)(U'\delta_k+u_k')(U'\delta_l+u_l')} \quad \text{(B 1)}$$

with the notation†

$$\delta_i = \begin{Bmatrix} 1, & i = 1 \\ 0, & i \neq 1 \end{Bmatrix}, \quad \delta_{ij} = \delta_i\delta_j, \quad \text{etc.}$$

reads, in expanded form:

$$\begin{aligned}\overline{v_i v_j v_k' v_l'} = {} & \overline{u_i u_j u_k' u_l'} + U(\delta_i\overline{u_j u_k' u_l'}+\delta_j\overline{u_i u_k' u_l'}) \\ & + U'(\delta_k\overline{u_i u_j u_l'}+\delta_l\overline{u_i u_j u_k'}) + U^2\delta_{ij}\overline{u_k' u_l'} + U'^2\delta_{kl}\overline{u_i u_j} \\ & + UU'(\delta_{ik}\overline{u_j u_l'}+\delta_{jl}\overline{u_i u_k'}+\delta_{jk}\overline{u_i u_l'}+\delta_{il}\overline{u_j u_k'}) + U^2U'^2\delta_{ijkl}, \quad \text{(B 2)}\end{aligned}$$

upon noting that $\overline{u_i} = \overline{u_j} = \overline{u_k'} = \overline{u_l'} = 0$ by definition. The terms in U^2, U'^2 and $U^2U'^2$ are constant with τ and are not sources of sound; they will be eliminated by the $\partial^4/\partial\tau^4$ operation of (10).

In order to deal with correlations like $U\overline{u_j u_k' u_l'}$ we first postulate that $\overline{u_j u_k' u_l'}$ is factorable into a function of τ (time delay) and a function of $\mathbf{r}$ (cf. (17)). Then, setting $\tau = 0$ after applying $\partial^4/\partial\tau^4$ as specified in the text, leaves (10) as merely a spatial integral. Next we reverse the co-ordinate transformation (4), reverting to the original co-ordinates $\mathbf{y}'$ and $\mathbf{y}''$ appearing in (3). Thus $d^3\mathbf{y}'$ replaces $d^3\mathbf{y}$ in (6) and $d^3\mathbf{y}''$ replaces $d^3\mathbf{r}$ in (7) and (10) for these correlations. The value of

† These special symbols, although somewhat similar, are not to be confused with Kronecker deltas.

$P(\theta, \phi)$ will be unaltered thereby. Since U refers to point $\mathbf{y}'$, it is invariant as $\mathbf{y}'$ is held fixed while $\mathbf{y}''$ is varied in performing

$$\int U\, \overline{u_j u_k' u_l'}\,(\mathbf{y}', \mathbf{y}'')\, d^3\mathbf{y}''.$$

For the assumed homogeneous isotropic turbulence this reduces to the form

$$U(\mathbf{y}') \int \overline{u_j u_n' u_l'}(\mathbf{y}'' - \mathbf{y}')\, d^3(\mathbf{y}'' - \mathbf{y}'),$$

and the volume integral of the triple correlation vanishes (Batchelor 1953).

Altogether, then, those correlations (designated by the symbol $\int =$) contributing to the noise integral are contained in

$$\overline{v_i v_j v_k' v_l'} \int = \overline{u_i u_j u_k' u_l'} + U U' (\delta_{ik} \overline{u_j u_l'} + \delta_{jl} \overline{u_i j_k'} + \delta_{jk} \overline{u_i u_l'} + \delta_{il} \overline{u_j u_k'}). \qquad \text{(B 3)}$$

Of these, the correlations contributing to the net radiation are those with indices as selected in (12) or (13). Insertion of these values of $ijkl$ gives the nine basic self and cross-correlations of (15). Permissible numbers of permutations of $ijkl$ are tabulated to be used as weight factors.

Appendix C. Relative magnitude shear noise and self noise

From (26) and (30) we may form the ratio

$$\frac{B}{A} = \frac{4\sqrt{(2\sigma)}}{(1+\sigma)^{\frac{3}{2}}} \frac{U^2}{\overline{u_1^2}} \frac{(g)^{\text{iv}}}{(g^2)^{\text{iv}}} = \frac{\text{peak shear noise}}{\text{self noise}}. \qquad \text{(C 1)}$$

For evaluation of B/A in the middle of the mixing region (radially) we have the following considerations. The turbulence level there is well known experimentally: $(\overline{u_1^2}/U^2) \simeq (0\cdot28)^2$ at $y/D = 4$ according to Laurence (1956). The value of σ depends on the turbulence scale L; experimentally the specification $\pi\sigma y^2/L^2 = 39\cdot0$ gives a good fit of (34) to data derived from Townsend's (1956, p. 177) mixing profile. If we identify L as the longitudinal scale, the experimental data for $y \simeq 4D$ show a broad spread, e.g.

	L/y	σ	$\sigma/(1+\sigma)^{\frac{3}{2}}$
Davies *et al.* (1963)	0·13	0·21	0·158
Laurence (1956)	0·075	0·070	0·063
Chu (1966)	0·0478	0·0284	0·0278

TABLE 1

The value $\sigma = 0\cdot45$ adopted by Ribner (1963, 1964) appears to be considerably in error.

For the further evaluation of B/A in (C 1) we require the ratio

$$\frac{(g)^{\text{iv}}}{(g^2)^{\text{iv}}} = \left[\frac{(\partial^4/\partial\tau^4)\, g(\tau)}{(\partial^4/\partial\tau^4)\, g^2(\tau)}\right]_{\tau=0} \qquad \text{(C 2)}$$

of fourth derivatives of the time correlation factor $g(\tau)$ and its square. Since for $k = i$ in (17) $g(\tau)$ plays the role of an autocorrelation, it must be an even function and it must possess (and $g^2(\tau)$ as well) a non-negative Fourier cosine transform. It is not clear how or if these restrictions on the form of $g(\tau)$ may limit the ratio $(g)^{\text{iv}}/(g^2)^{\text{iv}}$ that may be assumed. Several assumptions (with σ from table 1) and their consequences for the ratio B/A are as follows:

	$g(\tau)$	$(g)^{\text{iv}}/(g^2)^{\text{iv}}$	σ	B/A	Remarks
(1)	$\exp(-\omega_f\|\tau\|)$	$\frac{1}{16}$	0·45	1·16	Ribner (1963, 1964)
(2)	$\exp(-\omega_f^2\tau^2)$	$\frac{1}{4}$	0·070	1·15	Gaussian correlation. Scale from Laurence (1956)
(3)	$\exp(-a_2\tau^2-a_4\tau^4)$	1/2·4 approx.	0·0284	0·84	Experimental correlation and scale (Chu 1966)

TABLE 2

Chu himself obtained a value of $B/A = 2{\cdot}63$ from a fuller use of his experimental data, using considerably less restrictive assumptions than those herein; the value of B/A involved the ratio, however, of two experimentally derived fourth derivatives and is thus open to question.

Leaving aside Case 1 for the moment, it would seem that reasonable choices for $g(\tau)$, with other parameters derived from experiment,† give values of B/A (= peak shear noise/self noise) of the order of unity.

The referee has pointed out that Case 1, used in the author's previous work, is quite invalid. The function $e^{-\omega_f|\tau|}$ possesses a cusp at the origin so that its derivatives are undefined there. (The computed ratio of $(g)^{\text{iv}}/(g^2)^{\text{iv}}$ refers instead to $g = e^{-\omega_f\tau}$.) But the value $\sigma = 0{\cdot}45$ chosen is now seen to disagree markedly with experiment. By coincidence the errors are compensating: thus the prediction in Ribner (1963, 1964) that B/A is of the order of unity appears fortuitously to have been about right.

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† In (2) the most probable value is chosen; in (3) internal consistency governs.

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